Midsemestral examination 2009
B.Math.Hons. IIIrd year
Algebraic Number Theory : B.Sury

You may use auxiliary results, but quote them precisely. Be brief !

Q 1.
Prove that an algebraic integer all of whose conjugates lie on the unit circle, must be a root of unity.

OR

If θ is a non-zero algebraic integer and $K = \mathbf{Q}(\theta)$, show that the norm of the principal ideal (θ) equals $|N_{K/\mathbf{Q}}(\theta)|$.

Q 2.
Show that any ideal $I \neq 0$ in a Dedekind domain can be generated by two elements, one of which can be an arbitrary non-zero element.

OR

Show that a Dedekind domain which is a UFD must be a PID.

Q 3.
Let $K = \mathbf{Q}(\alpha)$ where α is a root of $X^3 + X + 1$. Prove that $\mathcal{O}_K = \mathbf{Z}[\alpha]$.

Q 4.
Prove the 'cyclotomic reciprocity law' : a prime p splits completely in $K := \mathbf{Q}[\zeta_n]$ (that is, $p\mathcal{O}_K = P_1 \cdots P_{\phi(n)}$) if, and only if, $p \equiv 1 \bmod n$.

OR

Let d be a square-free (positive or negative) integer. Find the decomposition of the prime 2 in the ring of integers of $\mathbf{Q}(\sqrt{d})$.

Q 5.
If K is a number field and p is a prime such that $p\mathbf{O}_K = P_1^{e_1} \cdots P_g^{e_g}$ where P_i's are prime ideals, prove that the norm of each P_i is of the form p^{f_i} and that we have $\sum_{i=1}^{g} e_i f_i = n$.

Q 6.
Let K denote the unique subfield of $\mathbf{Q}[\zeta_{31}]$ of degree 6 over $\mathbf{Q}$. Prove that $\mathcal{O}_K$ cannot be of the form $\mathbf{Z}[\alpha]$ for any α.
(*Hint :* Use the relation between "decomposition of p and decomposition of f mod p" along with the fact that if $\mathcal{O}_K$ were of the form $\mathbf{Z}[\alpha]$, then the minimal polynomial must factor modulo 2 as a product of linear factors.)